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Large size planar discrete velocity models for gas mixtures

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Abstract

We present results for the construction of physically acceptable large size planar discrete velocity models (DVMs), with momenta tiling all integer coordinates of the plane, for binary gas mixtures. We want, with binary collisions, five conservation laws (four for the restriction along one axis): only one for the light mass species with mass $m = 1$, only one for the heavy mass species with mass $M > 1$, only two for the momenta along the x, y axes and one for the energy. We restrict our study to models with one zero momentum, while the momenta in the plane are different and occupy only integer coordinates.

We start with a preliminary simple physical model satisfying the above constraints and add, with geometrical tools, new momenta. As an illustration we construct models, for $M = 2, 3, 4, 5$, with an arbitrary number of momenta and which occupy all integer coordinates of the plane.

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1. Introduction

We present physical discrete velocity models (DVMs) for binary planar gas mixtures (masses $m = 1$ and $M = 2, 3, 4, 5$ for the light and heavy species) with momenta tiling all the integer coordinates of the plane.

This is the outcome of recent interest [1, 7] occurring for the construction of physical binary DVM mixtures with only planar binary collisions and only a finite number of velocities (or momenta). First, the pioneering work was done by Bobylev and Cercignani [1] who gave two simple models, symmetric with respect to an exchange between the two axes: $13v_i$, $25v_i$ with $M = 2, 5$. Second, for the $13v_i$ models (Cornille and [3]), the restriction along one axis with seven densities and only two collisions explains easily why they have spurious (not physical) invariants. For the $25v_i$, $M = 2$, 5 models, it was mentioned [2, 4, 7] that other people with powerful computers, have also found spurious invariants. At that time (HC), two semisymmetric (symmetric with respect to the two axes but not to an exchange between them) models $11v_i$, $M > 1$, $13v_i$, $M = 5$ and one symmetric (we add another semisymmetric rotated by $\pi/2$ model) $17v_i$, $M = 2$ model were the first physical models found. Third, Cercignani and Cornille [3] studied shock waves for these two first physical models. Fourth, Bobylev and Cercignani [4] explained that for a physical model including a collision with three known momenta, we can add another. For binary collisions, the five conservation laws are satisfied and if three densities belong to the previous model, in order to eliminate this collision, we must add the last one. They said in conclusion that their study confirms that their previous [1] $25v_i$, $M = 2$, 5 models have spurious invariants.

Fifth, Cornille and Cercignani [5, 6] generalized their previous classes of physical semisymmetric $11v_i$, $13v_i$, $15v_i$ (with a minimal $9v_i$) and symmetric $17v_i$ models. They tried to detect geometrically some virus leading to the existence of spurious invariants. For instance, for the species without the zero momentum, we must have collisions connecting the momenta with $x \gtrless 0$ (also $y \gtrless 0$). Otherwise, we will have spurious invariants for the mass of this species. This type of virus was observed for both the semisymmetric $11v_i$, $M = 3$ and the $15v_i$, $M = 5$ models (leading to the $25v_i$ model [1]) but not for the $11v_i$, $M = 2$ and $15v_i, M = 2$. Then, in contast with [2, 4, 7] and results found with powerful computers, it was proved [5, 6] that the $25v_i$, $M = 2$ model [1] is physical. The explanation is simple. The $25v_i$ models have for $M = 2$, 5 common collisions leading to spurious invariants. A fine analysis (with more heavy calculations), explains geometrically why the $M = 2$ (in contrast to $M \neq 2$), has other collisions which eliminate these spurious invariants. Another virus, for the momentum invariant of a symmetric model exists when the associated semisymmetric is without momenta along the bisectors $y = \pm x$. Another drawback arises when a model, called 'ambiguous', has only five invariants but two for one species. In all models of [1, 6] there is a zero momentum and all densities have different momenta.

In contrast, in [7], physical models without a zero momentum and two densities having the same momenta were presented, except for a minimal $10v_i$ model with $M/m = 3$. But in this model, all momenta of one species are along one axis, the other species has two invariants ('ambiguous' model).

Finally the physical minimal and maximal models were the $9v_i$ of [5, 6] and the $25v_i$, $M = 2$ of [1, 5, 6]. Here we adopt a different point of view, with a presentation of $M = 2$, 3, 4, 5 physical models tiling, with momenta, the square lattice. The starting point is the determination of a model with few velocities (avoiding the different virus tested in [5, 6]) and we give, in appendices B–D, the complete proofs that it is physical. In the text of sections 3–5, we give, for the masses of the light and heavy species, more pedagogical proofs (explained in section 2). In section 2, we present *geometrical tools* (for any M), giving examples where we can extend a previous physical model, adding one or four or more new momenta. We explain that when the velocities of the light and heavy species are equal, then they cannot collide. In sections 3–5, we apply these geometrical tools and momenta fill all integer coordinates of the (x, y) plane. We call $(f_i, \boldsymbol{v}_i, \vec{p}_i, l_i)$, $(F_i, \boldsymbol{V}_i, \vec{P}_i, L_i)$ the densities, velocities, momenta and evolution equations of the light and heavy species, in particular (f_0, l_0), (F_0, L_0) for $\vec{p}_0 = (0, 0)$, $\vec{P}_0 = (0, 0)$. We recall that for a mixed collision $f_iF_j - f_kF_l$, we have $\vec{p}_i + \vec{P}_j = \vec{p}_k + \vec{P}_l$ and $|\vec{p}_i|^2 + |\vec{P}_j|^2/M = |\vec{p}_k|^2 + |\vec{P}_l|^2/M$, while for instance $|\vec{p}_i|^2 + |\vec{p}_j|^2 = |\vec{p}_k|^2 + |\vec{p}_l|^2$ for the light species. For any set a_i we define $a_{i,j,\ldots,p} = \sum_{s=i}^{s=p} a_s$. For the preliminary model, eliminating the collisions, we consider linear combinations of three sets: l_i, L_i and l_i, L_i (except l_0, L_0). We must prove for the masses that only $\mathcal{M}_l = \sum l_i = 0$, $\mathcal{M}_L = \sum L_i = 0$. The last set must be a linear combination of the three other physical invariants: momenta $\mathcal{J}_x$, $\mathcal{J}_y$ and energy $\mathcal{E}$.

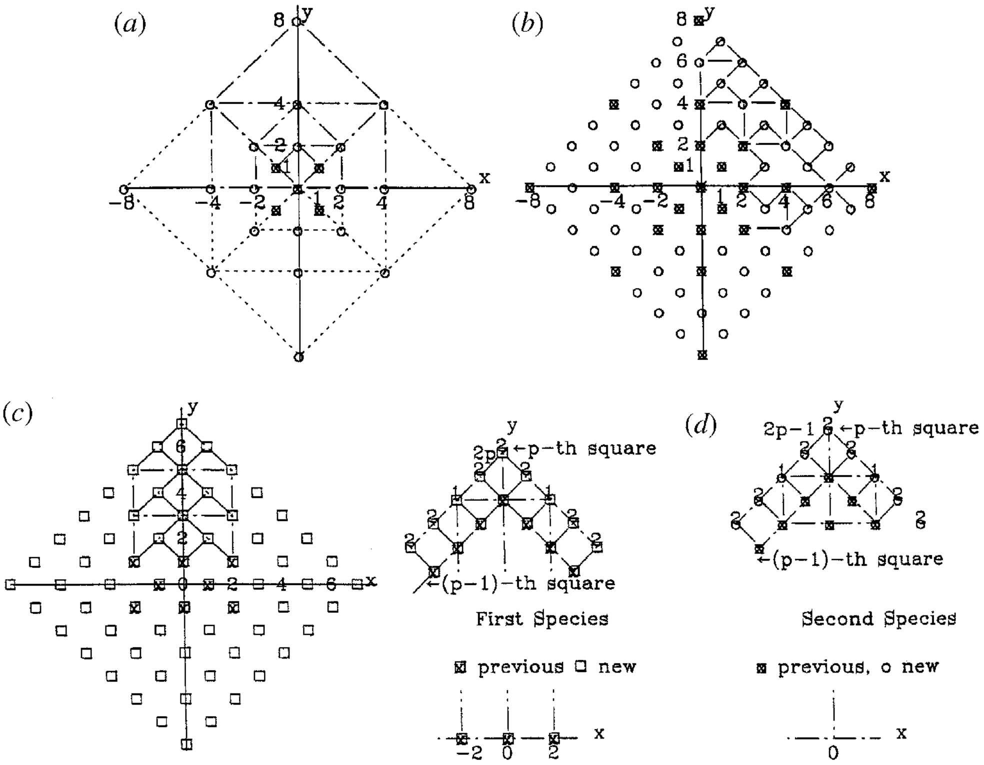

Figure 1. (*a*) Nested-squares for a single gas starting with 5: ⊠ previous, ○ new. (*b*) Single gas, squares $|x| + |y|$ even: ⊠, previous nested-squares, ○ new. (*c*) One species, squares $|x| + |y|$ odd: ⊠ previous, □ new. (*d*) Models satisfying lemmas 5–6.

2. Lemmas for geometrical planar tools to extend physical models

(1) For collisions of the same species, present in rectangles or squares of the momentum (x, y) plane, to the three known we can add another.

(2) For mixed collisions along rectangles with momenta of the same species symmetric with respect to a bisector and three known, we can add the last.

(3) For a light (or heavy) square, belonging to a physical mixture model, with centre $(0, 0)$ and four momenta along the diagonals (either the bisectors $y = \pm x$ or the x, y axes), we can add the four momenta along the medians and the new mixture model is physical (proof in appendix A).

(4) For a gas (not mixture), starting with $9v_i$:$(0, 0)$, $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 2, \pm 1)$, $(\pm 1, \mp 1)$, adding squares and rectangles, all integer coordinates are filled. We assume n integer and $(n - 1, 0)$, $(n, \pm 1)$ known (true for $n = 1, 9v_i$), add $(n + 1, 0)$ and $(n + 1, \pm 1)$ with $(0, 0)$, $(0, \pm 1)$. All integers of the x-axis (similarly for the y-axis) are filled. From $(0, 0)$ and any $(p, 0)$, $(0, q)$, we deduce (p, q).

(5) For a light (or heavy) species of a mixture, we start from a $\mathbf{5}v_i$ (or $\mathbf{5}V_i$) square $(0, 0)$, $(\pm 1, \pm 1)$, $(\mp 1, \pm 1)$, add with squares, $(\pm 2, 0)$, $(0, \pm 2)$ and get a $\mathbf{9}v_i$ (or $\mathbf{9}V_i$).

(i) Figure 1(a). Adding successively one new momentum from squares including $(0, 0)$ we deduce the nested-squares [8]: $(\pm 2^q, 0)$, $(0, \pm 2^q)$, $(\pm 2^q, \pm 2^q)$, $(\pm 2^q, \mp 2^q)$ $q = 1, 2, 3, \ldots$.
(ii) Figure 1(b). Adding squares, (not necessarily with $(0, 0)$), we get all integers $|x| + |y|$ even.

(6) Figure 1(c). We start with $\mathbf{8}\boldsymbol{v}_i$ (or $\mathbf{8}\boldsymbol{V}_i$): $(\pm 1, 0), (0, \pm 1)$, $(\pm 2, 1), (\pm 2, -1)$, $|x| + |y| = 1, 3$ and deduce all integer coordinates with $|x| + |y|$ odd.

(7) Figure 1(d): for models satisfying lemmas 5 and 6, alternatively, $|x| + |y|$ even or odd, the momenta of the first and second species belong to squares with diagonals along the x and y axes. For $M = 2$, we have found two such models.

First, for the first species with f_0 (F_0), the pth integer square has edges limited by $(\pm 2p, 0)$, $(0, \pm 2p)$. Starting with a previous known $(p - 1)$th square, we begin (i) with $(0, 0)$, $(\pm 2, 0), (0, 2(p - 1))$ and get two new momenta of the pth square $(\pm 2, 2(p - 1))$ (similarly for $y < 0$ and $x \gtrless 0$). (ii) We go on with squares, having edges parallel and perpendicular to the bisectors and we fill all momenta of the pth square. Second the pth square of the second species (without f_0 or F_0) is limited by $(0, \pm(2p - 1))$, $(\pm(2p - 1), 0)$. Starting with the $p - 1$, $p - 2$ squares and either squares parallel and perpendicular to the x, y or $y = \pm x$ axes, we get lemmas 1, 2, 8 and all momenta of the pth square.

(8) We assume, for a mixture with mixed collisions, that either the light species with f_0 (or heavy with F_0), has four invariants coming from collisions with only $\vec{p}_i$ (only $\vec{P}_i$). A mixed collision satisfies the conservation laws for the mixture, but in general not (except for the mass) for one species alone. For the (l_i), only $\mathcal{M}_l = \sum l_i = 0$, because $\mathcal{E} = 0$ cannot be satisfied with $f_0 F_i - f_j F_k$, similarly for $\mathcal{J}_x = \mathcal{J}_y = 0$ if $\vec{p}_j{:}(x \neq 0, y \neq 0)$ or with other mixed collisions.

(9) For any d-dimensional space and any M, then $\vec{p}_1 = (p_{1,1}, \ldots, p_{1,n})$ and $\vec{P}_1 = M\vec{p}_1$ (or $\boldsymbol{v}_1 = \boldsymbol{V}_1$) cannot collide. We define $\vec{A} = \vec{p}_1 + \vec{P}_1 = (M + 1)(p_{1,1}, \ldots, p_{1,n})$ and $2\mathcal{E}_1 = (\vec{p}_1)^2 + (\vec{P}_1)^2/M = (M+1)\sum_{i=1}^{d} p_{1,i}^2$. We consider arbitrary $\vec{p}_2 = (p_{2,1}, \ldots, p_{2,n})$ and associate $\vec{P}_2$ for a collision with $\vec{p}_1$, $\vec{P}_1$. Then $\vec{A} = \vec{p}_2 + \vec{P}_2$ and $2\mathcal{E}_2 = (\vec{p}_1)^2 + (\vec{P}_2)^2/M$, deduce $\vec{P}_2 = ((M+1)p_{1,1} - p_{2,1}, \ldots, (M+1)p_{1,d} - p_{2,d})$ from the $\vec{A}$ equalities and $2\mathcal{E}_2 = \sum_{i=1}^{d} [p_{2,i}^2 + (1/M)((M + 1)p_{1,i} - p_{2,i})^2]$. We get $0 = 2(\mathcal{E}_2 - \mathcal{E}_1) = ((M + 1)/M)\sum_1^d (p_{1,i} - p_{2,i})^2 = 0$ or $\vec{p}_1 = \vec{p}_2$, $\vec{P}_1 = \vec{P}_2$.

3. $M = 2$ models, figures 2(a)–(c), f_0

We construct two models including either f_0 or F_0, with momenta of both species along parallels to the bisectors $y = \pm x$. For the species with the zero momentum, the other momenta have $|x| + |y|$ even (odd for the other species).

3.1. First $M = 2$, f_0, figure 2(a) model, appendix B: $11v_i = 5\boldsymbol{v}_i + 6\boldsymbol{V}_i$

(i) The $11v_i$ model is physical [5, 6] (complete proof in appendix B). Here with lemma 8 we prove that $\mathcal{M}_l = 0 = \mathcal{M}_L$ (cf equation B.1). The five $\boldsymbol{v}_i$ light species has four invariants but with a collision $f_0 F_1 - f_1 F_5$, three disappear and only $\mathcal{M}_l = 0$. For $\mathcal{M}_L$, with collisions including f_0, we have two triplets $L_{j,j+2,j+4}$, $j = 1, 2$, but with $f_1 F_6 - f_2 F_5$, only $\mathcal{M}_L = 0$ remains. Adding $\vec{P}_7$, $\vec{P}_8$: $(0, \pm 1)$, the $13v_i = 5\boldsymbol{v}_i + 8\boldsymbol{V}_i$ model, figure 2(a), is also physical.

(ii) Starting with the five $\boldsymbol{v}_i$ (eight $\boldsymbol{V}_i$) species of the $11v_i$ ($13v_i$) models and with lemma 5(6), we fill all integer coordinates with momenta $|x| + |y|$ even (odd) and, figure 2(c), with lemma 7, all integer coordinates of the plane.

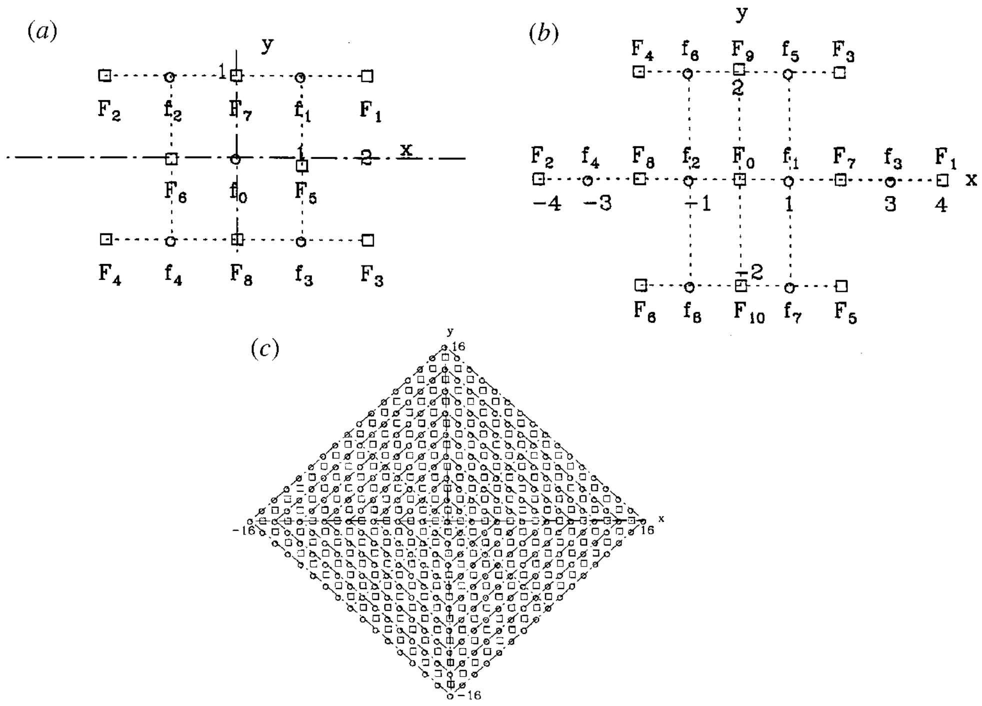

Figure 2. $M = 2$, f_0: □ heavy, ○light, 11v without, 13v with F_7, F_8. (*b*) $M = 2$, F_0, 15v and 19v. (*c*) First $M = 2$ model: ○light, □ heavy, f_0. (*c*) Dual $M = 2$ model: ○ heavy, □ light, F_0.

3.2. 'Dual' $M = 2$, F_0, figure 2(b) model: $15v_i = \mathbf{8v_i} + \mathbf{7V_i}$

(i) We start with the semisymmetric $15v_i$, $19v_i$ physical models. First, the $15v_i$ model, with eight $\vec{p}_i$ momenta: $(\pm 1, 0)$, $(\pm 3, 0)$, $(\pm 1, 2)$, $(\pm 1, -2)$ and seven $\vec{P}_i$:$(0, 0)$, $(\pm 4, 0)$, $(\pm 2, \pm 2)$, was very important because, adding the $\pi/2$ rotated model, it leads to the $25v_i$, $M = 2$ model [1]. With an analytical proof [5,6] (similar to appendix (B.1)), then (in contrast to what was written in the literature [2,4,7]) both $15v_i$ and $25v_i$ are physical models. Second, to the five $\vec{P}_i$, $i = 0, \ldots, 4$ we add four $\vec{P}_j$:$(0, \pm 2)$, $(\pm 2, 0)$, $j = 7, 0.10$ along the medians and (lemma 3), the $19v_i = \mathbf{8v_i}+\mathbf{11V_i}$ model is physical. Here, for brevity, we give only an independent proof for the mass of the heavy $\mathbf{11V_i}$ species which has four invariants but, with $F_0 f_5 - f_2 F_3$ (lemma 8), only $\mathcal{M}_L = 0$ remains. Third, we add the momenta symmetric with respect to the bisectors: two $\vec{P}_j$:$(0, \pm 4)$, eight $\vec{p}_i$:$(\pm 2, 1)$, $(\pm 2, -1)$, $(0, \pm 3)$, $(0, \pm 1)$, and with lemma 1, the $29v_i = 16v_i + 13\mathbf{V_i}$ model is physical.

(ii) The physical $29v_i$ has for light species the eight momenta of lemma 6: $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 2, 1)$, $(\pm 2, -1)$, then all integer coordinates with $|x| + |y|$ odd are filled.

(iii) We start with the square $\vec{P}_i$:$(0, 0)$, $(0, \pm 2)$, $(\pm 2, 0)$ (diagonals along the x, y axes) and (lemma 3) add the four $\vec{P}_j$ of the medians: $(\pm 1, 1)$, $(\pm 1, -1)$. Then from the $5\mathbf{v_i}$ of lemma 5 we see that all heavy momenta with $|x| + |y|$ even are filled.

(iv) Finally, figure 2(*c*), all integer coordinates of the plane are occupied.

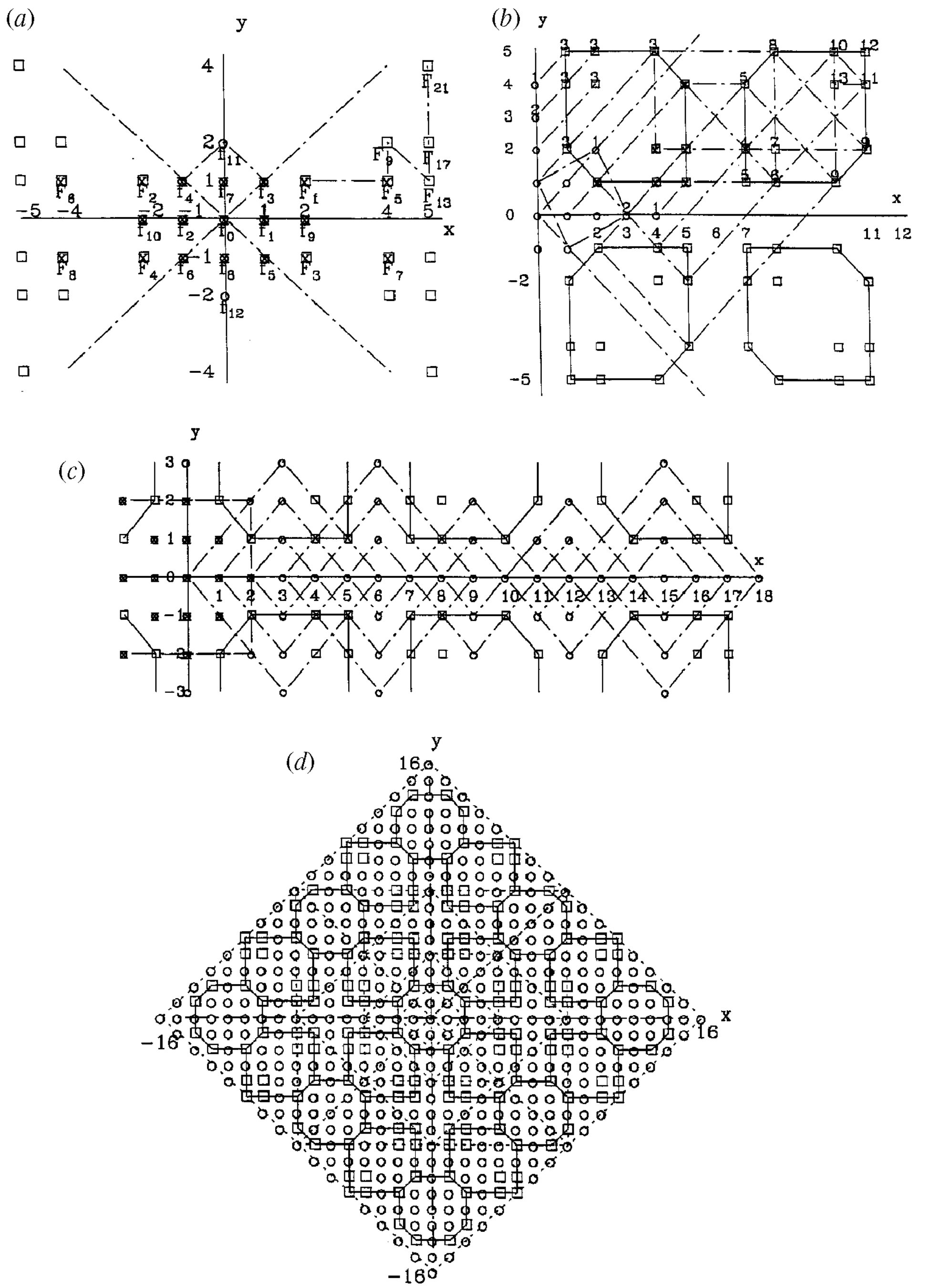

Figure 3. (*a*) $M = 3$, 19v and 37v, f_0, ⊠ heavy, ⊛ light. (*b*) $M = 3$, f_0, ○ light, ⊠ previous heavy, □ new heavy. (*c*) $M = 3$, f_0. □ heavy, ⊛ previous light, ○ new light. (*d*) $M = 3$ model: ○ light, □ heavy.

4. M = 3 and 5 models with f_0 (appendix C)

4.1. $19v_i = \mathbf{11v_i} + \mathbf{8V_i}$ *models, figures 3(a), 4(a)*

The momenta are written down in appendix C and it is proved that the $19v_i$, $M = 3, 5$ models are physical. The $\vec{p}_i$ and four $\vec{P}_i$ are the same for $M = 3,\ 5$, while $\vec{P}_j,\ j = 5, 6, 7, 8$, are different: same y but $x = \pm 4,\ \pm 3$, for $M = 3,\ 5$.

(i) Here we give an independent proof only for the masses. The same $\mathbf{11v_i}$ light species has four invariants but only one remains for the mixture (lemma 8). From the collisions equation (C.1): $f_0F_5 - \hat{f}F_1,\ f_5F_1 - f_4F_3$, we deduce that only $\mathcal{M}_l = 0$. For the heavy species we have four doublets $L_{j,j+2},\ j = 1, 2, 3, 4$ from the collisions including f_0, two quadruplets $L_{j,j+2,j+4,j+6},\ j = 1, 2$ from $F_if_8 - F_{i+2}f_7,\ i = 1, 2$ and finally $\mathcal{M}_L = \sum L_i$ from $F_1f_{10} - F_2f_9$.

(ii) For the set (l_i, L_i) except l_0, we prove, in appendix (C.1) that the invariants are a linear combination of the physical invariants $\mathcal{E},\ J_x,\ \mathcal{J}_y$.

(iii) Finally in C.1 we add $\vec{p}_i$:(0, ±2) leading to a $21v_i$ physical models.

4.2. $M = 3$ *model, figure 3(a)*

In appendix (C.2), to the $21v_i$ model, we add 16 $\vec{P}_j$ satisfying energy exchange collisions and the $37v_i$ model is physical.

Figure 3(b). We generalize the model, mainly with heavy momenta. (1) $\vec{p}_i$ (2, ±2) and (4, 0) from (0, 0), (0, ±2), (2, 0) and (0, 0), (2, ±2). (2) $\vec{p}_i$:(3, 0) from (2, 2), (0, 1), (1, −1). (3) We have $\vec{p}_j$:($x = 2, 3, 4,\ y = 0$), similarly ($x = 0,\ y = 2, 3, 4$) and with lemma 2 for mixed collisions, in the $x > 0,\ y > 0$ quadrant, we add six new $\vec{P}_j$ giving a first pattern (three others with the (x, y) symmetries). With $\vec{P}_j$ squares or rectangles, we get the fifth heavy pattern with new heavy momenta (called **4-5-6-7-8-9-10-11-12-13**) and we extend into the plane.

Figure 3(c). We introduce new $\vec{p}_i$ and begin with integer coordinates of the x-axis. With $\vec{p}_i$: $(x - a, 0), a = 1, 2, 3$ and $(x, \pm a)$ (if they are not occupied with $\vec{P}_i$), we get another $\vec{p}_i$:$(x+a, 0)$. Starting with $x = 0, 1, 2, 3, 4$ we get $x = 5, 6, 7, \ldots$, the whole x-axis (similarly y-axis) and, with f_0, all empty sites of the plane.

4.3. $M = 5$ *model, figure 4(a)*

In C.3 we add eight $\vec{P}_j$: $(\pm 4, 2\eta), (\pm 4, 3\eta), \eta = \pm 1$ satisfying energy exchange collisions and the $21v_i$ model becomes $29v_i$. With mixed rectangles (f_i, F_i) and lemma 2, we add four $\vec{P}_i$ ($x > 0,\ y > 0$) giving (with x, y symmetries) the $45v_i$ physical model (four heavy octagons).

Figure 4(b). For the extension of the number of heavy octagons, we consider rectangles with $\vec{P}_j$. We write successively **1-2-3-4-5-6-7-8** the new heavy momenta leading to a new octagon. With the x, y symmetries of the $45v_i$ model we get eight new octagons and we extend in the whole plane.

Figure 4(c). We show how empty sites (without $\vec{P}_j$) can be occupied by $\vec{p}_i$. For the x-axis, any integer $\vec{p}_i$ is the fourth unknown of light species rectangles (or squares). With $\vec{p}_i$:$(x, 0)$, $(0, \pm a), a = 1, 2,\ \vec{p}_0$ we get a new $\vec{p}_i$:$(x, \pm a)$. The squares with $(p - a, 0), (p, \pm a)$ give a new $(p + a, 0)$, except for $x = 3, 8, 13, \ldots, = 3 + 5q$ which are obtained from $(5q, 1), (5q + 1, -1), (5q + 2, 2)$ (similarly for the y-axis). Finally with $\vec{p}_0$ and these $\vec{p}_i$:$(x, 0), (0, y)$ we fill the plane (figure 4(d)).

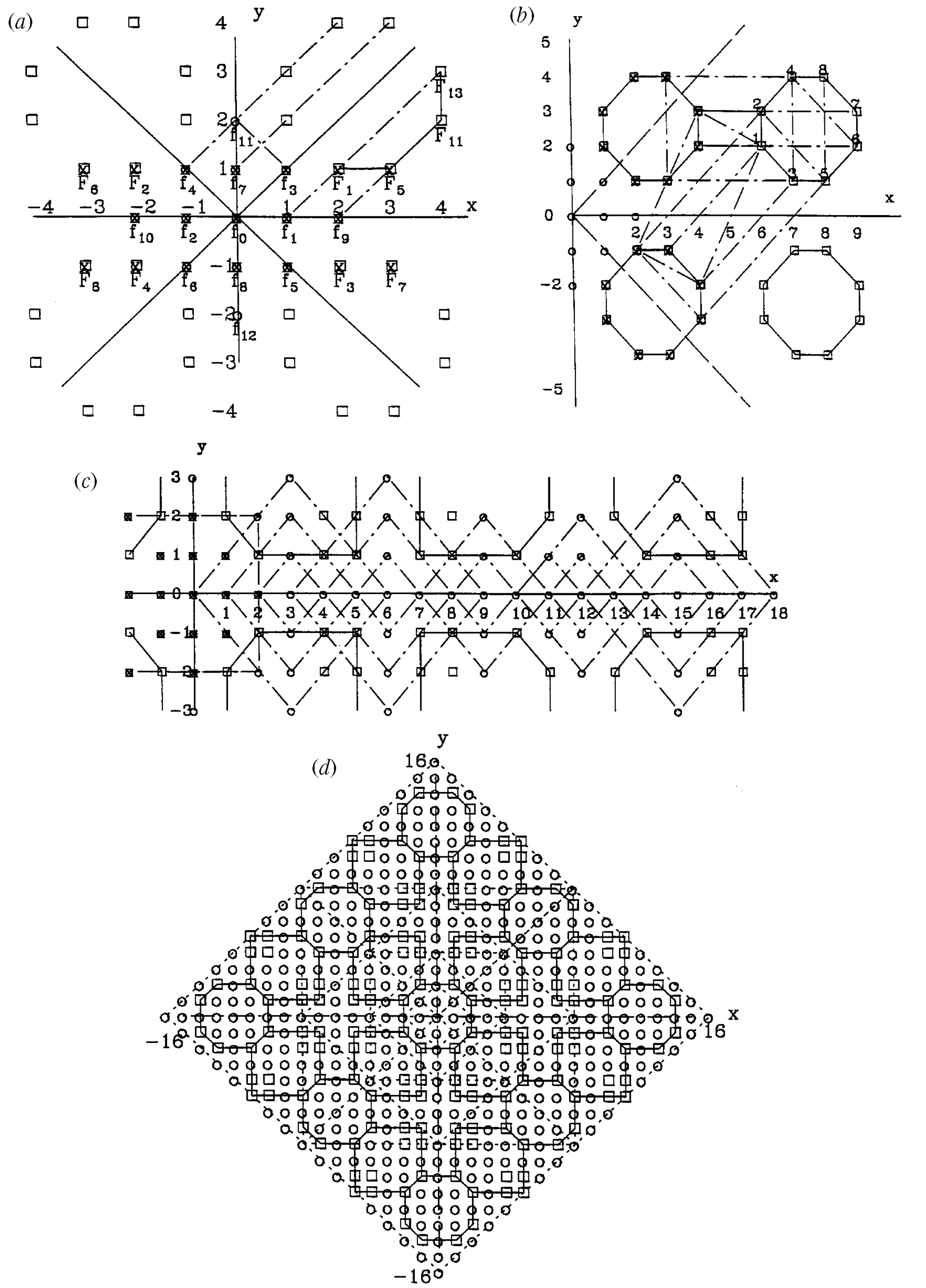

Figure 4. (*a*) $M = 5$, 19v, 29v and 45v, f_0: ⊠ heavy, ⊛ light, 19v. (*b*) $M = 5$, f_0: ○ light, ⊠ previous heavy, □ new heavy. (*c*) $M = 5$, f_0: □ heavy, ⊛ previous light, ○ new light. (*d*) $M = 5$ model: ○ light, □ heavy.

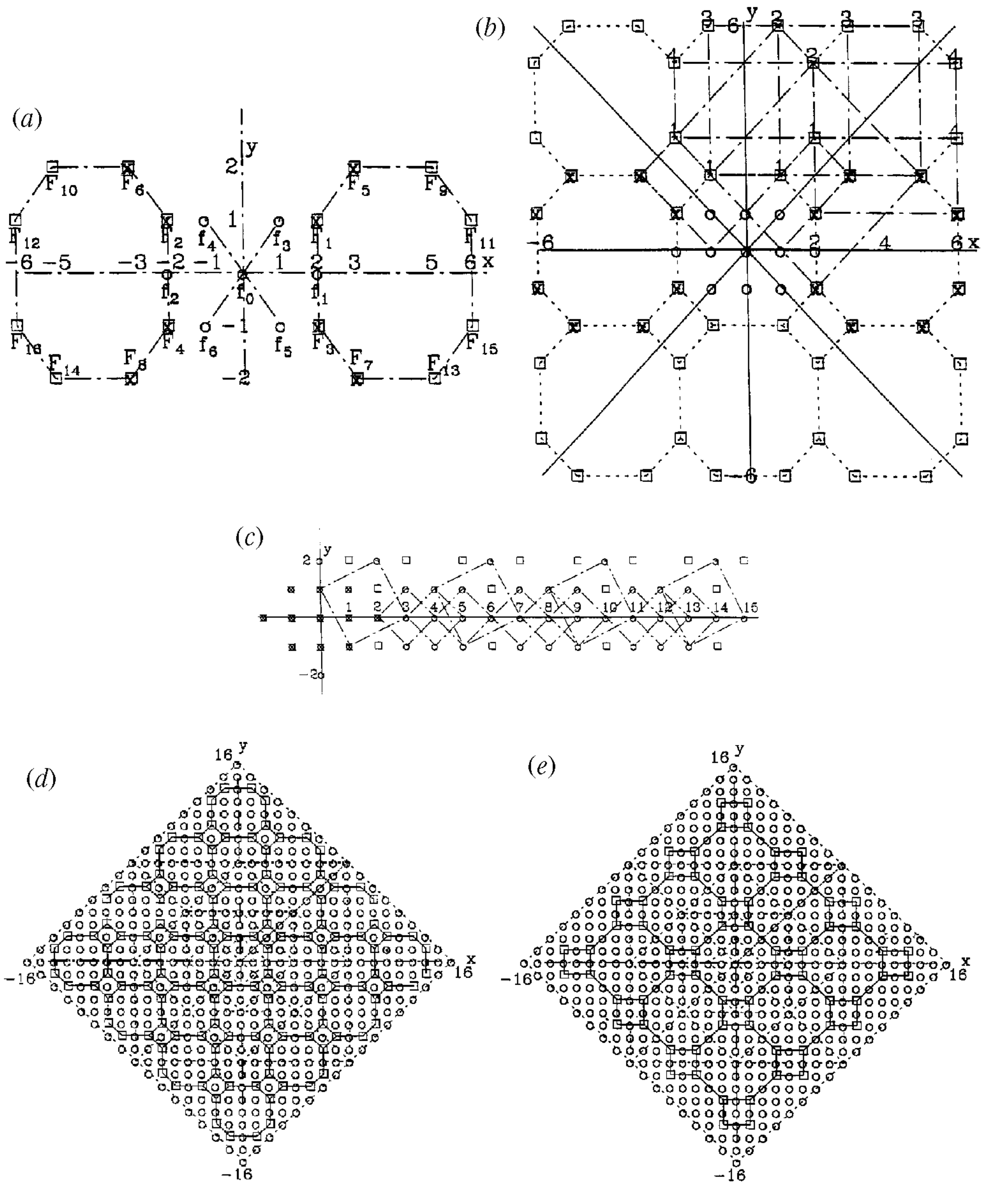

Figure 5. $M = 4$, 15v and (15 + 8)v with F_i, $i = 9, \ldots 16$: ⊠ previous heavy, ○ light, f_0. (*b*) $M = 4$, f_0: ⊠ previous heavy, □ new heavy, ○ light. (*c*) $M = 4$, f_0: ○ new light, ⊛ previous light, □ heavy. (*d*) $M = 4$ model: ○ light, □ heavy. (*e*) $M = 4$ model: ○ light, □ heavy.

5. M = 4 model with f_0 (appendix D), figures 5(*a*)– (*e*)

Figure 5(*a*). (i) $15v_i = \mathbf{7}\boldsymbol{v}_i + \mathbf{8}\boldsymbol{V}_i$ model with all densities f_i, F_j connected by collisions including f_0. To the complete proofs (appendix D with $\vec{p}_i$, $\vec{P}_j$, and collisions written down), we add lemma 8 for the masses. For the $\mathbf{7}\boldsymbol{v}_i$ with four invariants and mixed collisions (cf D1) like $f_0F_5 - f_3F_1$, three invariants disappear and only $\mathcal{M}_l = \sum l_i = 0$ remains. For the $\mathbf{8V_i}$, the collisions with f_0 give four doublets: $L_{i,i+4}, i = 1, 2, 3, 4$, two quadruplets $L_{i,i+1,i+4,i+6}, i = 1, 3$ (with collisions $F_i f_2 - F_{i+1} f_1, i = 1, 3$ linking doublets with $x \gtrless 0$) and only $\mathcal{M}_L = \sum L_i = 0$. with $F_1 f_5 - F_3 f_3$ linking $y \gtrless 0$.

(ii) $23v_i = \mathbf{7v_i} + \mathbf{16V_i}$ physical model with two heavy octagons. In appendix (D.2), we add eight $\vec{P}_i$ satisfying collisions with energy exchange and get a $23v_i = \mathbf{7v_i} + \mathbf{16V_i}$ physical model.

Figure 5(*b*). (i) In the square with $\vec{p}_0$ and diagonals $(\pm 1, \pm 1), (\pm 1, \mp 1)$ belonging to $y = \pm x$, we apply lemma 3 and add four $\vec{p}_j$:$(0, \pm 1), (\pm 1, 0)$ along the medians leading to $27v_i$.

(ii) We extend the two heavy octagons: **1** from rectangles with mixed collisions satisfying lemma 2, from rectangles with three known $\vec{P}_j$, **2-3-4** from rectangles parallel either to the $y = \pm x, x, y$ axes. Performing the same procedure for $x \gtrless 0$, $y \gtrless 0$, we get nine octagons and so on.

Figure 5(*c*). We show that all integer coordinates (without $\vec{P}_j$) can be filled with $\vec{p}_i$. We begin with $\vec{p}_i$:$(x > 0, 0)$ and $\vec{p}_i$ squares. We can obtain the nested-squares [8]: $(2^q, 0)$ or $x = 1, 2, 4, \ldots$, but many x values are missing. $(3, 0)$ is deduced from $(2, 2), (0, 1), (1, -1)$ and we add $(3, \pm 1)$. More generally, from $(x, 0)$ and $(0, 0)$, $(0, \pm 1)$ or $(0, \pm 2)$, we add $(x, \pm 1), (x, \pm 2)$ except if $\vec{P}_j$ are present. We get $(x, 0), x = 4, 5, 6$, but we are stopped in $(7, 0), (11, 0), \ldots, (3 + 4p, 0), p = 1, 2, 3, \ldots$ because $\vec{P}_j$:$(2 + 4p, \pm 1)$ are present. These $(3 + 4p, 0)$ momenta belong to $\vec{p}_i$:$(2 + 4p, 2), (4p, 1), (1 + 4p, -1)$. We can fill all integers $(x > 0, 0)$ (similarly $x < 0$, $y \gtrless 0$) and with $\vec{p}_0$, all empty sites are filled, figure 5(*d*).

Figure 5(*e*). We present another $M = 4$ physical model, still with $\vec{P}_j$ octagons, but different momenta locations. For brevity we do not give the proofs, but explain the construction of the physical preliminary model:

(i) With $\vec{p}_1, \vec{P}_1, \vec{P}_3$: $(2, 0), (3, 1), (5, 1)$, $\vec{p}_0$, and a collision $f_0 F_3 - f_1 F_1$, we get a semisymmetric $11v_i$ model. As recalled [5, 6] here, in order to avoid a spurious invariant for $\mathcal{M}_L$, we must include collisions connecting $x \gtrless 0$ (also $y \gtrless 0$). We add $\vec{p}_i$:$(\pm 3, 0), (0, \pm 1)$, leading to a $15v_i$ model. Concerning $\mathcal{M}_l$ and $\mathcal{J}_x, \mathcal{J}_y$, we must add $\vec{p}_i$ and momenta along $y = \pm x$. With ten $\vec{p}_i$: $(\pm 1, 0), (\pm 2, \pm 1), (\pm 2, \mp 1), (\pm 1, \pm 1), (\pm 1, \mp 1)$ we get a $25v_i$ physical model.

(ii) With squares we add $\vec{p}_i$:$(0, \pm 2), (\pm 2, \pm 2)$ and $\vec{P}_5, \vec{P}_7$: $(7, 3), (7, 5)$, from collisions with f_0 linking $F_i, F_{i+2}, i = 3, 5$. We have a semisymmetric $39v_i$ model.

(iii) With the section 2 lemmas we extend successively the $\vec{P}_i$ and $\vec{p}_i$ and fill all the plane (figure 5(*e*)).

6. Concluding remarks

In this paper we have presented 'physical' DVMs for mixtures with $M = 2, 3, 4, 5$ and momenta tiling all integers of the plane (the previous maximal physical [1, 5, 6] model was $25v_i, M = 2$). For their constructions we follow the same steps: (i) a collision with energy exchange including f_0 or F_0 and with symmetries $x \to -x, y \to -y$, the construction of semisymmetric models. (ii) If necessary, the addition of other momenta to eliminate spurious invariants (complete proofs in the appendices). (iii) Then with the geometrical lemmas of section 2, we construct the associated complete models, adding new $\vec{p}_i$, $\vec{P}_j$.

For M integer fixed, we can find different complete models, for instance here for $M = 2$ and 5. Can we construct complete models with noninteger values? We recall that in [3, 5, 6] we have studied different classes of $9v_i$, $11v_i$, $13v_i$, $15v_i$ semisymmetric models, associated with different starting collisions with energy exchange. For illustration we recall only one of them: $f_0F_i(p+q,\nu) - f_j(q,0)F_k(p,\nu)$, with p, q, ν integers leading to $M = 1 + 2p/q$, here the $M = 3$, 5 models with $(p,q,\nu) = (2,2,1), (2,1,1)$. More generally, we can get M rational for collisions with f_0 (F_0), but in order to avoid spurious invariants we must also satisfy (ii)–(iii). We have found such a complete model, not presented here, with $M = 3/2$, f_0 and octagons for the $\vec{P}_j$.

We have chosen $m = 1$ and M for $\vec{p}_i = \boldsymbol{v}_i$ and $\vec{P}_j = M\boldsymbol{V}_j$. The important parameter is the ratio $M/1$. If we choose $m = 2$ and $2M$, for the light and heavy masses, the ratio is still $M/1$ but $\vec{p}_i \to 2\vec{p}_i$, $\vec{P}_j \to 2\vec{P}_j$, giving the same dilated patterns for the complete models but with only the even coordinates occupied. Then the problem is whether we can also fill the odd empty sites.

In all our complete models we have, for the species with f_0 or F_0, the nested-squares [8] with $x = 2^q = 1, 2.4, \ldots, q = 0, 1, 2, \ldots$ along the $x > 0$ semi-axis (except $x = 1$ for $M = 2$) and $y = x$ along the bisector. There is a dissymmetry (due to our restriction of integer coordinates) between $x \to \infty$ and $\to 0$. The nested-squares were extended to $x = 2^{-q} = 1/2, 1/4, 1/8$, for a gas (not mixture). With lemma 3 (we can add four momenta along the medians from those along the diagonals), we can generalize the $M = 3, 4, 5$ presented models to new physical ones, but the coordinates are not always integers.

Appendix A. Lemma 3: from a $5V_i$ to a $9V_i$ subset of one species

(A.1): *Diagonals along the $y = \pm x$ axes.* We start with a $5V_i$ subset of a heavy species (same proof for a light), belonging to a physical mixture model: F_j, $\vec{P}_j$:(0, 0), $(\pm x, x)$, $i = 0, 1, 2$, $(\pm x, -x)$, $i = 3, 4$. We add $\vec{P}_j$:$(\pm x, 0)$, $j = 5, 6$, $(0, \pm x)$, $j = 7, 8$ and write only seven new collisions for the new mixture:

$$\begin{aligned}&\Gamma_i = F_0F_i - F_7F_{4+i} \quad i = 1,2 \qquad \Gamma_j = F_0F_j - F_8F_{2+j} \quad j = 3,4\\&\Gamma_5 = F_1F_6 - F_2F_5 \qquad \Gamma_6 = F_1F_8 - F_3F_7 \qquad \Gamma_7 = F_5F_6 - F_7F_8.\end{aligned} \tag{A.1}$$

To the old mass $\hat{\mathcal{M}}_L = \cdots + \sum_0^4 L_i = -2\Gamma_{1,2,3,4}$ and energy $\hat{\mathcal{E}} = \cdots + x^2L_{1,2,3,4} = -x^2\sum_1^4\Gamma_i$, the only possible new ones are $\mathcal{M}_L = \hat{\mathcal{M}}_L + \sum_5^8 L_i = 0$ and $\mathcal{E} = \hat{\mathcal{E}} + (x^2)/2)\sum_5^8 L_i$. Eliminating Γ_i, $i = 5, 6, 1, 2, 3, 4$, we get $\hat{\mathcal{M}}_L + aL_{5,6} + (2-a)L_{7,8} = 4\Gamma_7(1-a) = 0 \to a = 1$ and $0 = \hat{\mathcal{E}} + aL_{5,6} + (x^2 - a)L_{7,8} = 2\Gamma_7(x^2 - 2a) \to a = x^2/2$. To the old momentum $\hat{\mathcal{J}}_x = \cdots + x(L_{1,3} - L_{2,4}) = x(\Gamma_{2,4} - \Gamma_{1,3} - 2\Gamma_5)$, the new one is $\mathcal{J}_x = \hat{\mathcal{J}}_x + xL^-_{5,6}$ (similarly $\mathcal{J}_y = \hat{\mathcal{J}}_y + xL^-_{7,8}$). Eliminating Γ_6, Γ_7 we get: $\hat{\mathcal{J}}_x + aL_5 + bL_6 + (a+b)L_{7,8}/2 = (a - b - 2x)\Gamma_5 + \Gamma_{2,4}(x + (3b+a)/2) + \Gamma_{1,3}(-x + (3a+b)/2) = 0 \to x = a = -b \to \mathcal{J}_x = 0$. For the L_i (except L_0), eliminating Γ_i, $i = 1, 2, 3, 4, 7$ (independent proof), we get new invariants (a, b, c const), linear combinations of physical ones:

$$\begin{aligned}&\mathcal{J}_y = \cdots + x\overline{\mathcal{J}}_y \qquad \overline{\mathcal{J}}_y = L_{1,2,7} - L_{3,4,8} \qquad \mathcal{J}_x = \cdots + x\overline{\mathcal{J}}_x \qquad \overline{\mathcal{J}}_x = L_{1,3,5} - L_{2,4,6}\\&2\mathcal{E} = \cdots + x^2\overline{\mathcal{E}} \qquad \overline{\mathcal{E}} = 2\sum_1^4 L_i + \sum_5^8 L_i\\&X = aL_1 + cL_2 + (2a + c - 4b)L_3 + (2c + a - 4b)L_4 + (a - b)L_5\\&\qquad +(c - b)L_6 + bL_7 + (a + c - 3b)L_8 = aA + bB + cC = 0\\&2C = \overline{\mathcal{E}} - \overline{\mathcal{J}}_y - \overline{\mathcal{J}}_x = 0 \qquad 2A = \overline{\mathcal{E}} - \overline{\mathcal{J}}_y + \overline{\mathcal{J}}_x = 0 \qquad B = -\overline{\mathcal{E}} + 2\overline{\mathcal{J}}_y = 0.\end{aligned} \tag{A.2}$$

(A.1bis): From a $\mathbf{5V}_i$ to a $\mathbf{11V}_i$ subset. We add $\vec{P}_j$:$(0, \pm 2x)$, F_j, $j = 9, 10$ and with $F_0F_9 - F_1F_2$, $F_0F_{10} - F_3F_4$, the new model is physical.
(A.2): Same results, with a rotation of $\pi/2$ for the diagonals along the x, y axes.

Appendix B. Physical $M = 2$, $f_0 5v_i + 6V_i, 5v_i + 8V_i$ models, figure 2(*a*)

$\mathbf{5v}_i + \mathbf{6V}_i$, $\vec{p}_i$:$(0, 0), (\pm 1, 1), (\pm 1, -1), i = 0, 0.4$, $\vec{P}_j$:$(\pm 2, 1), (\pm 2, -1), (\pm 1, 0)$, $j = 1, \ldots .$ 6 mixed collisions are sufficient (five for the masses) for the proofs:

$$\begin{aligned} &\Gamma_i = f_0F_i - f_iF_{4+i} && i = 1, 2 \\ &\Gamma_j = f_0F_j - f_jF_{2+j} && j = 3, 4 \\ &\Gamma_5 = f_1F_6 - f_2F_5 && \Gamma_6 = f_3F_6 - f_4F_5. \end{aligned} \tag{B.1}$$

For the $(l_i), i = 0, \ldots, 4$ and the $(L_i), i = 1, \ldots, 6$, we eliminate successively $\Gamma_i, i = 1, 3, 5, 2, 4$ and the only possible linear combinations are $\mathcal{M}_l = \sum l_i = 0$ and $\mathcal{M}_L = \sum L_i = 0$. For the set (l_i, L_i) except l_0), the only new three invariants must be: the energy $\mathcal{E} - \mathcal{M}_L/4 = L_{1,2,3,4} + l_{1,2,3,4} = \overline{\mathcal{E}}$ and the momenta:

$$\mathcal{J}_y = L_{1,2} - L_{3,4} + l_{1,2} - l_{3,4} \qquad \mathcal{J}_x = 2(L_{1,3} - L_{2,4}) + L_5 - L_6 + l_{1,3} - l_{2,4}. \tag{B.2}$$

We eliminate Γ_1 with $l_1 + aL_5 + (1 + a)L_1$, and so on for Γ_i, $i = 5, 2, 4, 3, 6$. We find a linear combination with three constants:

$$\begin{aligned} X - a\mathcal{M}_L &= l_1 + L_1 + bl_2 + (b - 1)L_6 + (2b - 1)L_2 + cl_4 + (c + 1 - b)l_3 \\ &\quad + (c + 1 - b)L_3 + (b + c - 1)L_4 = 0 = D + bB + cC = 0 \\ 2D = \mathcal{J}_x + \overline{\mathcal{E}} - \mathcal{M}_L = 0 &\qquad 2B = \mathcal{M}_L + \mathcal{J}_y - \mathcal{J}_x = 0 \qquad 2C = \overline{\mathcal{E}} - \mathcal{J}_y = 0 \end{aligned} \tag{B.3}$$

($\mathbf{5v}_i + \mathbf{8V}_i$): We add $\vec{P}_j$:$(0, \pm 1)$, $j = 7, 8$ and 2 collisions: $\Gamma_7 = f_1F_8 - f_3F_7$, $\Omega = F_5F_6 - F_7F_8$. For the L_i, eliminating Γ_7, Ω we get $\sum_1^6 L_i + dL_{7,8} = 2(d - 1)\Omega = 0$ or $\mathcal{M}_L = \sum_1^8 L_i = 0$. For the (l_i, L_i) except l_0, with $\overline{\mathcal{E}}$, $\mathcal{J}_x$ as above, we change: $\mathcal{J}_y = \cdots + L_7 - L_8$, $X - a\mathcal{M}_L = \cdots + (b - (c + 1)/2)L_7 + L_8(c - 1)/2 = 0$, and add $L_8 - L_7$, $-L_{7,8}$, L_7 to $2C$, $2D$, B. The relations (B.3) linking the new D, B, C to $\overline{\mathcal{E}}$, $\mathcal{J}_x$, $\mathcal{J}_y$ are still valid.

Appendix C. Physical, $M = 5$ and 3, models, figures 3(*a*), 4(*a*)

(C.1): $\mathbf{11v}_i + \mathbf{8V}_i$, $\vec{p}_i$:$(0, 0), (\pm 1, 0), (\pm 1, 1), (\pm 1, -1), (0, \pm 1), (\pm 2, 0), i = 0, \ldots, 10$, $\vec{P}_j$:$(\pm 2, 1), (\pm 2, -1), j = 1, \ldots, 4$ common to $M = 3, 5$. We add $j = 5, \ldots, 8$, $(\pm 4, 1), (\pm 4, -1)$ for $M = 3$ and $(\pm 3, 1), (\pm 3.0 - 1)$ for $M = 5$. We write mixed collisions with energy exchange Γ_j, $j = 1, \ldots, 4$ and 5 other with $j = 5, \ldots, 9$:

$$\begin{aligned} &\Gamma_j = f_0F_{j+4} - \hat{f}F_j : j = 1, 3 \qquad \hat{f} = f_9(M = 3) = f_1(M = 5) \\ &\text{and} \\ &j = 2, 4 \qquad \hat{f} = f_{10}(M = 3) = f_2(M = 5) \\ &\Gamma_5 = F_1f_{10} - F_2f_9 \qquad \Gamma_{5+i} = F_if_8 - F_{2+i}f_7 \\ &\Gamma_{7+i} = f_{4+i}F_1 - f_{2+i}F_3, \qquad i = 1, 2 \end{aligned} \tag{C.1}$$

(i) For the L_i, we eliminate successively: $\Gamma_i, i = 1, 6, 3, 5, 7, 2, 4$ and find only $\mathcal{M}_L = \sum L_i = 0$.

(ii) For the l_i, we introduce eight light collision terms:

$$\begin{aligned} &\Lambda_i = f_0 f_{8+i} - f_{2+i} f_{4+i} \qquad \Lambda_{2+i} = f_0 f_{2+i} - f_i f_7 \\ &\Lambda_{4+i} = f_0 f_{4+i} - f_i f_8 \qquad i = 1, 2 \qquad \Lambda_7 = f_1 f_2 - f_7 f_8 \\ &\Lambda_8 = f_3 f_2 - f_4 f_1. \end{aligned} \tag{C.2}$$

We eliminate $\Gamma_i, i = 5, 6, 8, 9, 1, 2$ and get three doublets $l_{7,8}, l_{3,5}, l_{4,6}$ and for $M = 3, 5$ a triplet $l_{0,1,2},\ l_{0,9,10}$. Then, with successively $\Lambda_3, \Lambda_4, \Lambda_1, \Lambda_2$, $M = 5$ and and $\Lambda_i, i = 1, \ldots, M = 3$ we get only $\mathcal{M}_l = \sum l_i = 0$.

(iii) For (l_i, L_i) except l_0, we eliminate the Λ_i and get a linear combination with constants $a,\ b$:

$$\begin{aligned} X = l_9 + al_3 + (1-a)l_5 + bl_1 + (1-3b)l_2 + (1+a-4b)l_4 + (2-a-4b)l_6 \\ +(1-a-b)l_8 + (3-8b)l_{10} + (a-b)l_7 \rightarrow X([\Lambda_i, i = 1, \ldots, 8]) = 0. \end{aligned} \tag{C.3}$$

For the elimination of the Γ_i, we distinguish between $M = 5$ and 3:

$$\begin{aligned} \mathcal{M} = 5 : Y = X + c\mathcal{M} + (1-2a)L_3 + bL_5 + (1-2a+b)L_7 \\ +(2-8b)L_2 + (3-11b)L_6 + (3-2a-8b)L_4 \\ +(4-2a-11b)L_8 = 0 = D + aA + bB \end{aligned} \tag{C.4a}$$

$$\begin{aligned} \mathcal{M} = 3 : Y = X + c\mathcal{M} + (1-2a)L_3 + L_5 + (2-2a)L_7 + (2-8b)L_2 \\ +(5-16b)L_6 + (3-2a-8b)L_4 \\ +(6-2a-16b)L_8 = 0 = D + aA + bB. \end{aligned} \tag{C.4b}$$

The $D = A = B = 0$ are linear combinations of $\overline{\mathcal{E}} = \mathcal{E}+$ const $\mathcal{M}_L,\ \mathcal{J}_x,\ \mathcal{J}_y$:

$$\begin{aligned} &2\overline{\mathcal{E}} = 2l_{3,4,5,6} + l_{1,2,7,8} + 4l_{9,10} + \nu L_{5,6,7,8} \\ &\mathcal{J}_y = L_{1,2,5,6} - L_{3,4,7,8} + l_{3,4,7} - l_{5,6,8} \\ &\mathcal{J}_x = L_{1,3} - L_{2,4} + l_{1,3,5} - l_{2,6,4} + 2(l_9 - l_{10} + \lambda(L_{5,7} - L_{6,8}). \\ &2D = 2\overline{\mathcal{E}} - \mathcal{J}_x - \mathcal{J}_y \qquad A = -M_L + \mathcal{J}_y \qquad B = 2(-\overline{\mathcal{E}} + \mathcal{J}_x). \end{aligned} \tag{C.5}$$

with $(\nu, \lambda) := (1, 3),\ (4, 4)$ for $M = 5,\ 3$. Finally, with $f_0 f_{11} - f_3 f_4$, we can add $\vec{p}_j(0, \pm 2),\ f_j,\ j = 11,\ 12$ and $\mathbf{11}\boldsymbol{v}_i \rightarrow \mathbf{13}\boldsymbol{v}_i$.

(C.2): $\mathbf{13}\boldsymbol{v}_i$**+**$\mathbf{24}\boldsymbol{V}_i$, $M = 3$ model, figure 3(a). We add four $\vec{P}_j$:(4, 2), (5, 1), (5, 2), (5, 4), $F_j,\ j = 9, 13, 17, 21$ (12 other with x, y symmetries) satisfying $f_0F_9 - f_7F_5,\ f_0F_{13} - f_5F_9,\ f_0F_{17} - f_7F_{13},\ f_0F_{21} - f_{11}F_{17}$.

(C.3) $\mathbf{13}\boldsymbol{v}_i$**+**$\mathbf{16}\boldsymbol{V}_i$, $M = 5$ model figure 4(a). We add two $\vec{P}_j$:(4, 2), (4, 3), $F_j,\ j = 9, 13$ (six others with x, y symmetries) satisfying $f_0F_9 - f_3F_5,\ f_0F_{13} - f_7F_9$.

Appendix D. Physical M = 4, 7v_i + 8V_i, 7v_i+16V_i models, figure 5(a)

(D.1): $\mathbf{7}\boldsymbol{v}_i$**+**$\mathbf{8}\boldsymbol{V}_i$, $\vec{p}_i$:(0, 0), (±2, 0), (±1, 1), (±1, −1), $i = 0, \ldots, 6$, $\vec{P}_j$:(±, 2), (±2, −1), (±3, 2), (±3, −2), $j = 1, \ldots, 8$. We write seven Γ_j and three Λ_i collisions:

$$\begin{aligned} &\Gamma_i = f_0 F_{4+i} - f_{2+i} F_i, i = 1, 2, 3, 4 \qquad \Gamma_5 = f_2 F_1 - F_2 f_1 \qquad \Gamma_6 = F_2 f_6 - F_4 f_4 \\ &\Gamma_7 = F_1 f_5 - F_3 f_3 \qquad \Lambda_i = f_0 f_i - f_{2+i} f_{4+i}, i = 1, 2 \qquad \Lambda_3 = f_3 f_6 - f_4 f_3. \end{aligned} \tag{D.1}$$

We write, as an illustration, less important collisions: $\Omega_i = F_i F_{i+3} - F_{i+1} F_{i+2}, i = 1, 5,\ \Gamma_8 = F_3 f_2 - F_4 f_1,\ \Gamma_{i,j} = F_i f_{j+2} - F_{i+2} f_j, (i, j) = (1, 4), (2, 3)$

(i) For the L_i, we eliminate successively Γ_i, $i = 1, 7, 3, 5, 2, 6, 4$ and find only $\mathcal{M}_L = \sum L_i = 0$. For the l_i, we eliminate Γ_5 with $l_{1,2} = -\Lambda_{1,2}$, Γ_i, $i = 1, 2, 6, 7$ with $l_{0,3,4,5,6} = \Lambda_{1,2}$ and Λ_i, $i = 1, 2$ giving only $\mathcal{M}_l = \sum_0^6 l_i = 0$.

(ii) For the l_i, L_j (except l_0), first we eliminate the three Λ_i and get a linear combination $X([\Lambda_i], a, b) = 0$ (two constants a, b). Second we eliminate successively the Γ_i, $i = 1, 2, 5, 3, 7, 6, 4$, (a third constant with $c\mathcal{M}_L = 0$):

$$X = l_1 + al_3 + (1 - a)l_5 + bl_2 + (a + (b - 1)/2))l_4 + ((b + 1)/2 - a)l_6.$$

$$Y - c\mathcal{M}_L = X + (b - 1)L_2 + (1 - 2a)L_3 + (b - 2a)L_4 + a)L_5 + (a + 3(b - 1)/2)L_6 \quad \text{(D.2)}$$
$$+(2 - 3a)L_7 + (1/2 + 3b/2 - 3a)L_8 = 0 = D + aA + bB.$$

The $D = A = B = 0$ are linear combinations of $\mathcal{J}_x, \mathcal{J}_y, \overline{\mathcal{E}} = \mathcal{E}+$ const $\mathcal{M}_L$.

$$\overline{\mathcal{E}} = L_{5,6,7,8} + l_{3,4,5,6} + 2l_{1,2}$$
$$\mathcal{J}_x = 3(L_{5,7} - L_{6,8}) + 2(L_{1,3} - L_{2,4}) \quad \text{(D.3)}$$
$$+l_{3,5} - l_{4,6} + 2(l_1 - l_2)\mathcal{J}_y = 2(L_{5,6} - L_{7,8}) + L_{1,2} - L_{3,4} + l_{3,4} - l_{5,6}$$

$$4D = \overline{\mathcal{E}} - J\mathcal{J}_x - 2\mathcal{J}_y) \qquad A = -\mathcal{M}_L + \mathcal{J}_y \qquad B = 2(\overline{\mathcal{E}} - \mathcal{J}_x)/4. \quad \text{(D.4)}$$

(D.2): $\mathbf{7v_i + 16V_i}$. We add $\vec{P}_j$:$(\pm 5, 2), (\pm 6, 1), (\pm 5, -2), (\pm 6, -1), F_j, j = 9, 10, \ldots, 16$, from collisions with f_0 (three known) like $f_0 F_9 - f_1 F_5,\ f_0 F_{11} - F_9 f_5$.

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